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Catenarity in quantum algebras [1]

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Abstract

Various quantum algebras are shown to be catenary, i.e., all saturated chains of prime ideals between any two fixed primes have the same length. Further, Tauvel's formula relating the height of a prime ideal to the Gelfand–Kirillov dimension of the corresponding factor ring is established. These results are obtained for coordinate rings of quantum affine spaces, for quantized Weyl algebras, and for coordinate rings of complex quantum general linear groups, as well as for quantized enveloping algebras of maximal nilpotent subalgebras of semisimple complex Lie algebras.

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0. Introduction

Our aim in this paper is to investigate the prime ideal structure of various algebras arising in the theory of quantum groups under the general heading "quantum coordinate rings". These algebras are expected to exhibit a structure similar to that of enveloping algebras of solvable Lie algebras, and our results support this philosophy. (See [3, 5, 9] for additional evidence.) Here, we concentrate on the property of catenarity. Recall that a ring R is *catenary* if, for any two prime ideals $P < Q$ of R, all saturated chains of prime ideals between P and Q have the same length. It is a famous result of Gabber that enveloping algebras of finite dimensional solvable Lie algebras are catenary (see, e.g., [7] or a combination of [18, Appendix A1] and [16, Ch. 9]); the second author has

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extended this result to enveloping algebras of finite dimensional solvable Lie superalgebras [17]. The method of proof in both cases is the same: establish good homological properties of the ring, then connect homological properties with growth, and, finally, control the growth properties of prime factors by finding normal elements. This method yields in addition the following useful height formula established by Tauvel [30] for enveloping algebras of solvable Lie algebras: $\text{height}(P) + \text{GKdim}(R/P) = \text{GKdim}(R)$ for all prime ideals P in R.

In Section 1, we start by writing down the abstract properties sufficient to prove catenarity results in this way, and in succeeding sections we show that various quantum algebras have these properties. In particular, we obtain catenarity and the height formula for coördinate rings of quantum affine spaces, for quantized Weyl algebras, and for coordinate rings of complex quantum general linear groups, as well as for quantized enveloping algebras of maximal nilpotent subalgebras of semisimple complex Lie algebras. (In the second and third cases, certain parameters are restricted to be nonroots of unity.)

1. Catenarity: Gabber's method abstracted

1.1. Throughout this section, let k be a field and R an affine (finitely generated) noetherian k-algebra. We show how existing technology can be combined to provide an axiomatic basis for Gabber's catenarity theorem. The homological aspects of this argument are contained in work of Björk, who has shown that a version of Gabber's maximality principle follows from the Auslander–Gorenstein condition [2]. The remainder of the proof runs parallel with the treatment of Gabber's theorem in [16], given the assumption of a suitable supply of normal elements.

The algebra R is said to be *Auslander–Gorenstein* provided (a) the injective dimension of R (as both a right and a left R-module) is finite, and (b) for any integers $0 \leq i < j$ and any finitely generated (right or left) R-module M, we have $\text{Ext}^i_R(N,R) = 0$ for all R-submodules N of $\text{Ext}^j_R(M,R)$. If, in addition, the global dimension of R is finite, then R is said to be *Auslander-regular*. The *grade* of a finitely generated R-module M is defined to be

$$j(M) := \inf\{j \geq 0 \mid \text{Ext}^j_R(M,R) \neq 0\}.$$

We shall denote the *Gelfand-Kirillov dimension* of an R-module M by $\text{GKdim}(M)$ (see [16 or 24, Ch. 8] for basic properties of this dimension). The algebra R is called *Cohen–Macaulay* (or *CM*) if $j(M)+\text{GKdim}(M) = \text{GKdim}(R)$ for all finitely generated R-modules M.

A nonzero R-module M is *pure* (with respect to grade) if $j(N) = j(M)$ for all nonzero submodules N of M. The corresponding property with grade replaced by GK-dimension is called *GK-homogeneity*. More precisely, M is called *s-homogeneous* if M and all its nonzero submodules have GK-dimension s.

Theorem 1.2 (Gabber's maximality principle). *Let R be Auslander–Gorenstein and M a finitely generated R-module. Suppose that M is pure, and that $\widetilde{M}$ is an R-module containing M such that every finitely generated submodule of $\widetilde{M}$ is pure. Then there is a unique maximal member in the set*

$$\{\text{finitely generated } R\text{-modules } X \mid M \subseteq X \subseteq \widetilde{M} \text{ and } j(X/M) \geq j(M) + 2\}.$$

Proof. See [2, Theorem 1.14]. □

Corollary 1.3. *Let R be Auslander–Gorenstein and Cohen–Macaulay, and M a finitely generated R-module. Suppose that* $\mathrm{GKdim}(R) < \infty$, *that M is GK-homogeneous, and that $\widetilde{M}$ is an R-module containing M such that every finitely generated submodule of $\widetilde{M}$ is GK-homogeneous. Then there is a unique maximal member in the set*

$$\{\text{finitely generated } R\text{-modules } Y \mid M \subseteq Y \subseteq \widetilde{M} \text{ and } \mathrm{GKdim}(Y/M) \leq \mathrm{GKdim}(M) - 2\}.$$

Proof. Translate the theorem from the setting of grade to that of GK-dimension using the CM hypothesis: $\mathrm{GKdim}(N) = \mathrm{GKdim}(R) - j(N)$ for all finitely generated R-modules N. □

If $P < Q$ are prime ideals of the noetherian k-algebra R with $\mathrm{height}(Q/P) = 1$, then $\mathrm{GKdim}(R/P) \geq \mathrm{GKdim}(R/Q) + 1$ [16, Corollary 3.16]. This inequality can be strict, and the next theorem, the key to catenarity, gives conditions sufficient to ensure equality.

Theorem 1.4. *Let R be Auslander–Gorenstein and Cohen–Macaulay with* $\mathrm{GKdim}(R)$ *finite, and let $P < Q$ be prime ideals of R with* $\mathrm{height}(Q/P) = 1$. *If there exists an element $x \in Q \setminus P$ that is normal modulo P, then* $\mathrm{GKdim}(R/P) = \mathrm{GKdim}(R/Q) + 1$.

Proof. Set $A = R/P$ and $s = \mathrm{GKdim}(A) < \infty$. Set $b = x + P \in A$, and note that $bA = Ab$ by hypothesis. Let $A[b^{-1}]$ be the localization of A with respect to the Ore set $\{b^n \mid n = 0, 1, 2, \ldots\}$. Now consider A and $A[b^{-1}]$ as right R-modules. Note that A is s-homogeneous by [16, Lemma 5.12], and that $A[b^{-1}]$ is the union of submodules $b^{-n}A = Ab^{-n}$, all of which are isomorphic to A (as one-sided modules). Hence, every finitely generated submodule of $A[b^{-1}]$ is s-homogeneous. Let Y be the unique maximal finitely generated extension of A in $A[b^{-1}]$ with $\mathrm{GKdim}(Y/A) \leq s - 2$ provided by Corollary 1.3. Since Y is finitely generated, there is an integer n such that $Y \subseteq b^{-n}A$. Now conjugation by b acts as an automorphism on both A and $A[b^{-1}]$, and it follows from the maximality of Y that $b^{-1}Yb = Y$; thus $bY = Yb$.

Let $I = \mathrm{r.ann}_A(Y/bY)$, and note that $I \neq 0$ (because $b \in I$). Then $YI \subseteq bY$ and so $YI^t \subseteq b^tY$ for each positive integer t. In particular,

$$I^{n+1} = AI^{n+1} \subseteq YI^{n+1} \subseteq b^{n+1}Y \subseteq b^{n+1}b^{-n}A = bA \subseteq Q/P,$$

and consequently $I \subseteq Q/P$. Thus Q/P is a prime ideal of A minimal over I.

Now $\mathrm{GKdim}(Y/bY) \le \mathrm{GKdim}(A/I) \le \mathrm{GKdim}(A) - 1 = s - 1$, by [16, Proposition 3.15]. On the other hand, $Y/bY \cong b^{-1}Y/Y$. Hence, by the maximality of Y, it follows that the module Y/bY is $(s-1)$-homogeneous. Now Lemmas 2 and 3 of [17] applied to Y/bY show that

$$\mathrm{GKdim}(R/Q) = \mathrm{GKdim}(A/(Q/P)) = s - 1 = \mathrm{GKdim}(R/P) - 1. \qquad \square$$

1.5. We say that the prime spectrum of the ring R has *normal separation* provided that for any pair of distinct comparable prime ideals $P < Q$ in $\operatorname{spec} R$, the factor Q/P contains a nonzero normal element of R/P. In the stronger setting where each factor Q/P contains a nonzero central element of R/P we say that $\operatorname{spec} R$ has *central separation*. Although we are not going to pursue questions about cliques, localization, and representation theory here, it is worth reminding the reader that normal separation is important for such questions in that it guarantees the *strong second layer condition* (cf. [14, Proposition 8.1.7; 10, Lemma 11.14]).

We shall say that *Tauvel's height formula* holds in the algebra R provided

$$\mathrm{height}(P) + \mathrm{GKdim}(R/P) = \mathrm{GKdim}(R)$$

for all $P \in \operatorname{spec} R$. In case R is in addition catenary, it follows that the height formula also holds in prime factor rings of R, since then

$$\mathrm{height}(Q/P) = \mathrm{height}(Q) - \mathrm{height}(P) = \mathrm{GKdim}(R/P) - \mathrm{GKdim}(R/Q)$$

for all primes $P < Q$ in R.

Theorem 1.4 allows us to deduce catenarity from normal separation together with the Auslander–Gorenstein and Cohen–Macaulay properties, as follows.

Theorem 1.6. *Let R be an affine, noetherian, Auslander–Gorenstein, Cohen–Macaulay algebra over a field, with finite Gelfand–Kirillov dimension. If* $\operatorname{spec} R$ *is normally separated, then R is catenary. If, in addition, R is a prime ring, then Tauvel's height formula holds.*

Proof. Suppose that $P = P_0 < P_1 < \cdots < P_n = Q$ is a saturated chain of prime ideals in $\operatorname{spec} R$. Then by Theorem 1.4,

$$\mathrm{GKdim}(R/P_{i-1}) - \mathrm{GKdim}(R/P_i) = 1$$

for each $i = 1, \ldots, n$. Summing up these equations, we find that

$$\mathrm{GKdim}(R/P) - \mathrm{GKdim}(R/Q) = n.$$

Therefore all saturated chains of prime ideals from P to Q have the same length, namely $\mathrm{GKdim}(R/P) - \mathrm{GKdim}(R/Q)$. Finally, in case R is a prime ring, we obtain Tauvel's height formula on taking $P = 0$. $\square$

2. Quantum affine spaces

Our first application is to coordinate rings of quantum affine spaces, where the Auslander-regular, Cohen–Macaulay, and normal separation properties are all relatively easy to verify. We also verify normal separation for certain localizations of these algebras, although that level of generality is not needed for our current applications.

2.1. Let $\lambda = (\lambda_{ij})$ be an $n \times n$ matrix of nonzero elements of a field k such that $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ij}^{-1}$ for $1 \leq i,j \leq n$. The multiparameter coordinate ring of quantum affine n-space is the k-algebra $\mathcal{O}_\lambda(k^n)$ generated by elements $x_1,\dots,x_n$ subject only to the relations $x_ix_j = \lambda_{ij}x_jx_i$ for $1 \leq i,j \leq n$. Note that $\mathcal{O}_\lambda(k^n)$ can be expressed as an iterated skew polynomial ring starting with the field k; hence, $\mathcal{O}_\lambda(k^n)$ is an affine noetherian domain. As in [23], we write $P(\lambda)$ for the localization of $\mathcal{O}_\lambda(k^n)$ with respect to the multiplicative set generated by the x_i, that is, $P(\lambda)$ is the k-algebra generated by $x_1,x_1^{-1},\dots,x_n,x_n^{-1}$ subject to the relations $x_ix_j = \lambda_{ij}x_jx_i$. We use standard multi-index notation for monomials in $\mathcal{O}_\lambda(k^n)$ and $P(\lambda)$: for any n-tuple $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$, we set $x^m = x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$.

2.2 An ideal I in a ring R is said to be *polynormal* (*polycentral*) provided I can be generated by a sequence of elements $c_1,\dots,c_t$ such that c_1 is normal (central) in R and c_i is normal (central) modulo $\langle c_1,\dots,c_{i-1}\rangle$ for $i = 2,\dots,t$. A *polynormal (polycentral) ring* is a ring in which all ideals are polynormal (polycentral).

Proposition. *For $l = 0,\dots,n$, let $A_l = k\langle x_1^{\pm 1},\dots,x_l^{\pm 1},x_{l+1},\dots,x_n\rangle$. Then A_l is polynormal, while $A_n = P(\lambda)$ is polycentral.*

Proof. Note that A_l is noetherian. Hence, it suffices to show that for any ideals $I > J$ in A_l, there exists an element $u \in I \setminus J$ such that $u + J$ is normal in A_l/J, and that $u + J$ can be chosen to be central in A_n/J in case $l = n$.

Define the *length* of an element $a \in A_l$ to be the number of distinct monomials x^m appearing in a with nonzero coefficients. Choose an element $u \in I \setminus J$ of minimum length, say length d. Then $u = \alpha_1x^{s_1} + \alpha_2x^{s_2} + \dots + \alpha_dx^{s_d}$ for some nonzero scalars α_h and some distinct n-tuples $s_h \in \mathbb{Z}^l \times (\mathbb{Z}^+)^{n-l}$. Since we may replace u by $\alpha_1^{-1}u$, there is no loss of generality in assuming that $\alpha_1 = 1$. Further, in case $l = n$ all the x_i are units in A_l, and so we can replace u by $x_n^{-s_{1n}}x_{n-1}^{-s_{1,n-1}}\cdots x_1^{-s_{11}}u$. Hence, in case $l = n$ we may assume that $s_1 = (0,\dots,0)$.

Fix $t \in \{1,\dots,n\}$. Each $x^{s_h}x_t = \beta_{ht}x_tx^{s_h}$ where $\beta_{ht} = \prod_{i=1}^n \lambda_{it}^{s_{hi}} \in k^\times$. Hence, $ux_t - \beta_{1t}x_tu = \alpha_2(\beta_{2t} - 1)x_tx^{s_2} + \dots + \alpha_d(\beta_{dt} - 1)x_tx^{s_d}$. This is an element of I with length less than d, and so it lies in J by minimality of d. Thus $ux_t \equiv \beta_{1t}x_tu \pmod J$ for all $t = 1,\dots,n$. Further, for $t = 1,\dots,l$ we have $ux_t^{-1} \equiv \beta_{1t}^{-1}x_t^{-1}u \pmod J$, and so $u + J$ is normal in A_l/J. In the case $l = n$, where we can assume that all $s_{1i} = 0$, we have $\beta_{1t} = 1$ for all t. Therefore $u + J$ is central in A_n/J in case $l = n$. □

2.3 (a) We can express $P(\boldsymbol{\lambda})$ as a quotient of a group algebra kG where G is the subgroup of the group of units of $P(\boldsymbol{\lambda})$ generated by the x_i and the λ_{ij}. Since G is finitely generated nilpotent, kG is polycentral by [27, Theorem A]. This provides an alternate proof of the polycentrality of $P(\boldsymbol{\lambda})$.

(b) A more detailed analysis of the structure of $P(\boldsymbol{\lambda})$ shows that any pair of primes $P > Q$ in $P(\boldsymbol{\lambda})$ can be separated by an element which is actually central in $P(\boldsymbol{\lambda})$ (rather than just central in $P(\boldsymbol{\lambda})/Q$). However, we shall not need this fact.

(c) We give a generalization of Proposition 2.2 in the following section (see Proposition 3.10).

Corollary 2.4. (a) $\operatorname{spec} P(\boldsymbol{\lambda})$ *has central separation.*

(b) *If A' is the subalgebra of $P(\boldsymbol{\lambda})$ generated by some subset of $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$, then* $\operatorname{spec} A'$ *has normal separation. In particular,* $\operatorname{spec} \mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ *has normal separation.*

Proof. (a) This is immediate from Proposition 2.2.

(b) After re-indexing, we may assume that A' is generated by

$$x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_s, x_{s+1}^{-1}, \ldots, x_t^{-1}$$

where $0 \le r \le s \le t \le n$. Then A' is a subalgebra of $P(\boldsymbol{\mu})$ where $\boldsymbol{\mu}$ is the upper left $t \times t$ block of $\boldsymbol{\lambda}$, and so after replacing $P(\boldsymbol{\lambda})$ by $P(\boldsymbol{\mu})$ we may assume that $t = n$. Secondly, we may rewrite $P(\boldsymbol{\lambda})$ as $P(\boldsymbol{\rho})$ with generators $x_1^{\pm 1}, \ldots, x_s^{\pm 1}, (x_{s+1}^{-1})^{\pm 1}, \ldots, (x_n^{-1})^{\pm 1}$ where

$$\rho_{ij} = \begin{cases} \lambda_{ij} & (i,j \le s \text{ or } i,j > s), \\ \lambda_{ij}^{-1} & (i \le s < j \text{ or } i > s \ge j). \end{cases}$$

Therefore without loss of generality, $s = n$, and the result follows from Proposition 2.2. □

The following properties of the algebra $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ are probably well known, but we have not located references in the literature.

Theorem 2.5. *The coordinate ring $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ of quantum affine n-space is Auslander-regular and Cohen–Macaulay, with GK-dimension n.*

Proof. Let V be the finite dimensional generating subspace for the algebra $A = \mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ spanned by $1, x_1, \ldots, x_n$. Since the monomials x^m, for $m \in (\mathbb{Z}^+)^n$, form a basis for A, we see that the dimension of each V^d equals the number of n-tuples $m = (m_1, \ldots, m_n)$ in $(\mathbb{Z}^+)^n$ with $m_1 + \cdots + m_n \le d$. These dimensions are the same as for a commutative polynomial ring in n variables, and thus $\operatorname{GKdim}(A) = n$ (cf. [16, Example 3.6] or [24, Proposition 8.1.15]).

We can write A as an iterated Ore extension in the form

$$A = k[x_1; \tau_1][x_2; \tau_2] \cdots [x_n; \tau_n],$$

where the τ_i are k-algebra automorphisms of the intermediate algebras

$$A_{i-1} := k[x_1;\tau_1][x_2;\tau_2]\cdots[x_{i-1};\tau_{i-1}],$$

such that $\tau_i(x_j) = \lambda_{ij}x_j$ for $i > j$. Further, if A_{i-1} is graded by total degree in $x_1,\ldots,x_{i-1}$, then A_{i-1} is connected and τ_i respects the grading. Iterated application of [19, Lemma] completes the proof. □

Theorem 2.6. *The coordinate ring $\mathcal{O}_\lambda(k^n)$ of quantum affine n-space is catenary, and Tauvel's height formula holds in $\mathcal{O}_\lambda(k^n)$.*

Proof. See Theorem 2.5, Corollary 2.4(b), and Theorem 1.6. □

3. Quantized Weyl algebras

In this section, we turn to the class of quantized Weyl algebras that first appeared in work of Maltsiniotis [22]. The Auslander-regular and Cohen–Macaulay properties are again not hard to verify; in fact, this has already been done by Giaquinto and Zhang [8] for a different class of quantized Weyl algebras that has a large intersection with the class we are interested in, and their methods work in our case as well. It is the verification of normal separation, which we obtain under the assumption that certain parameters q_i are not roots of unity, which absorbs most of our effort.

3.1. Let $A = A_n^{Q,\Gamma}(k)$ be a multiparameter quantized Weyl algebra over a field k as in [22] or [6, 12.5] (cf. [1, 5, 15]). Here $Q = (q_1,\ldots,q_n) \in (k^\times)^n$ and $\Gamma = (\gamma_{ij}) \in M_n(k^\times)$ with $\gamma_{ii} = 1$ and $\gamma_{ji} = \gamma_{ij}^{-1}$ for all i,j. The algebra A is generated by elements $x_1, y_1,\ldots,x_n, y_n$ subject to the following relations:

$$y_iy_j = \gamma_{ij}y_jy_i \quad (\text{all } i,j),$$
$$x_ix_j = q_i\gamma_{ij}x_jx_i \quad (i < j),$$
$$x_iy_j = \gamma_{ji}y_jx_i \quad (i < j),$$
$$x_iy_j = q_j\gamma_{ji}y_jx_i \quad (i > j),$$
$$x_jy_j = 1 + q_jy_jx_j + \sum_{l<j}(q_l - 1)y_lx_l \quad (\text{all } j).$$

It follows easily from these relations that A can be presented as an iterated skew polynomial ring of the form

$$k[y_1][x_1;\tau_2,\delta_2][y_2;\tau_3][x_2;\tau_4,\delta_4]\cdots[y_n;\tau_{2n-1}][x_n;\tau_{2n},\delta_{2n}],$$

where the τ_i are k-algebra automorphisms and the δ_{2i} are k-linear τ_{2i}-derivations (cf. [15, 2.1, 2.8]). In particular, A is thus seen to be an affine noetherian domain.

3.2. Let $A = A_n^{Q,\Gamma}(k)$ as in (3.1). Set $z_j = x_jy_j - y_jx_j = 1 + \sum_{l=1}^{j}(q_l - 1)y_lx_l$ for $j = 1,\ldots,n$; note also that $x_jy_j - q_jy_jx_j = z_{j-1}$ for $j \geq 2$. As shown in [15, 2.8], the

z_j are normal in A. More precisely,

$$z_j y_i = q_i y_i z_j, \quad z_j x_i = q_i^{-1} x_i z_j \qquad (i \le j)$$

$$z_j y_i = y_i z_j, \quad z_j x_i = x_i z_j \qquad (i > j)$$

$$z_j z_i = z_i z_j \qquad (\text{all } i, j).$$

In particular, the normality of the z_i allows us to localize with respect to any set of them. More precisely, given any $M \subseteq \{1, \dots, n\}$, the multiplicative set $\mathscr{Z}_M$ generated by $\{z_m \mid m \in M\}$ is an Ore set, and so there exists an Ore localization $A[\mathscr{Z}_M^{-1}]$. Moreover, the corresponding multiplicative sets $\mathscr{X}_M$ and $\mathscr{Y}_M$, generated by $\{x_m \mid m \in M\}$ and $\{y_m \mid m \in M\}$, respectively, are Ore sets, as observed in [15, 3.1].

We follow Jordan [15, 3.1] in denoting the localization $A[\mathscr{Z}_{\{1,\dots,n\}}^{-1}]$ by $B_n^{Q,\Gamma}(k)$.

We begin our analysis of quantized Weyl algebras by calculating the Gelfand–Kirillov dimensions of the localizations $A_n^{Q,\Gamma}(k)[\mathscr{Z}_M^{-1}]$. The GK-dimension of $A_n^{Q,\Gamma}(k)$ in the case where all the q_i coincide has been calculated by Giaquinto and Zhang [8, Theorem 3.11]. The following lemma, analogous to [16, Proposition 4.2; 24, Proposition 8.2.13], will be helpful.

Lemma 3.3. *Let A be an affine k-algebra with a regular normal element z. Suppose that A has a finite dimensional generating subspace V containing 1 such that $zV = Vz$. Then*

$$\operatorname{GKdim} A[z^{-1}] = \operatorname{GKdim} A.$$

Proof. Set $W = V + kz^{-1}$. Then W is a finite dimensional generating subspace for $A[z^{-1}]$ containing 1, and $Wz = zW \subseteq V^t$ for some t. Hence, $W^n \subseteq V^{nt} z^{-n}$ for each n, so that $\dim W^n \le \dim V^{nt}$. Thus $\operatorname{GKdim} A[z^{-1}] \le \operatorname{GKdim} A$. The opposite inequality is obvious since A is a subalgebra of $A[z^{-1}]$. $\square$

Proposition 3.4. *For any $M \subseteq \{1, \dots, n\}$, the algebra $A_n^{Q,\Gamma}(k)[\mathscr{Z}_M^{-1}]$ has GK-dimension $2n$.*

Proof. Set $A = A_n^{Q,\Gamma}(k)$. We first calculate the GK-dimension of A itself. Let V be the finite dimensional generating subspace of A spanned by $1, x_i, y_i, 1 \le i \le n$. Using the relations given in (3.1), it is easy to see that each V^d is spanned by monomials $y_1^{i_1} \cdots y_n^{i_n} x_1^{j_1} \cdots x_n^{j_n}$ of total degree less than or equal to d. Also, these monomials are linearly independent, since A is an iterated Ore extension in the variables y_i, x_i (3.1). Hence, $\dim V^d = \binom{2n+d}{2n}$, the same as for the polynomial ring in $2n$ variables (cf. [24, Lemma 8.1.3]). Thus $\operatorname{GKdim} A = 2n$.

Now set $z = \prod_{i \in M} z_i$ (this product may be taken in any order, since the z_i commute with each other). Then $A[\mathscr{Z}_M^{-1}] = A[z^{-1}]$. The relations given in (3.2) for the z_i show that if V is the generating subspace for A given above, then $zV = Vz$. That $\operatorname{GKdim} A[\mathscr{Z}_M^{-1}] = \operatorname{GKdim} A = 2n$ now follows from the previous lemma. $\square$

Giaquinto and Zhang have proved the Auslander-regular and Cohen–Macaulay properties for a class of quantized Weyl algebras that includes those $A_n^{Q,\Gamma}(k)$ with $Q = (q,q,\dots,q)$ [8, Theorem 3.11]. We derive these properties for general $A_n^{Q,\Gamma}(k)$ in a similar manner.

Lemma 3.5. *Filter the algebra $A = A_n^{Q,\Gamma}(k)$ by total degree in the x_i and y_i, and let C be the corresponding associated graded algebra. Let X_i, Y_i be the principal symbols of x_i, y_i, respectively, and for each $m = 0,\dots,n$ let C_m be the algebra generated by X_i and Y_i for $i = 1,\dots,m$. Then, C_m is an iterated skew polynomial algebra of the form*

$$C_m = C_{m-1}[Y_m;\tau_{2m-1}][X_m;\tau_{2m},\delta_{2m}].$$

Further, if C_{m-1} and $C_{m-1}[Y_m;\tau_{2m-1}]$ are graded by total degree, then τ_{2m-1} and τ_{2m} are graded algebra automorphisms.

Proof. The relations between the X_i and Y_j are the same as those between the x_i and y_j given in (3.1) except that the 1 is deleted from the last relation, i.e.,

$$X_jY_j = q_jY_jX_j + \sum_{l<j}(q_l - 1)Y_lX_l$$

for all j. Using these relations, it is easy to see that the monomials

$$(*) \qquad Y_1^{i_1}\cdots Y_n^{i_n}X_1^{j_1}\cdots X_n^{j_n}, \quad \sum_{s=1}^{n}(i_s + j_s) = d$$

span the dth homogeneous component V^d/V^{d-1} of C. Now

$$\dim V^d - \dim V^{d-1} = \binom{2n+d}{2n} - \binom{2n+d-1}{2n},$$

as in the proof of Proposition 3.4, and this is exactly the number of monomials of the form $(*)$; hence, these monomials must be linearly independent. Since C is the direct sum of its homogeneous components, the monomials Y^iX^j, for $i,j \in (\mathbb{Z}^+)^n$, are linearly independent.

Now the relations between X_i and Y_j for $i,j \le m-1$ do not involve $Y_m, Y_{m+1},\dots,Y_n$ or $X_m, X_{m+1},\dots,X_n$, and so the monomials $Y_1^{i_1}\cdots Y_m^{i_m}X_1^{j_1}\cdots X_m^{j_m}$ span C_m. Thus, $C_{m-1}\langle Y_m\rangle$ is an Ore extension of C_{m-1}, and inspection of the relations shows that it has the form $C_{m-1}[Y_m;\tau_{2m-1}]$ where τ_{2m-1} is a k-algebra automorphism respecting the grading on C_{m-1}. Similarly, C_m is an Ore extension of the form $C_{m-1}[Y_m;\tau_{2m-1}][X_m;\tau_{2m},\delta_{2m}]$, where τ_{2m} is a k-algebra automorphism respecting the grading on $C_{m-1}[Y_m;\tau_{2m-1}]$ and δ_{2m} is a k-linear τ_{2m}-derivation. □

Theorem 3.6. *The quantized Weyl algebra $A_n^{Q,\Gamma}(k)$ is an Auslander-regular Cohen–Macaulay algebra.*

Proof. Let $C = C_n$ be the associated graded algebra of $A = A_n^{Q,\Gamma}(k)$, where A is filtered by total degree. The previous lemma shows that C is an iterated skew polynomial algebra of the form

$$k[Y_1][X_1;\tau_2,\delta_2][Y_2;\tau_3][X_2;\tau_4,\delta_4]\cdots[Y_n;\tau_{2n-1}][X_n;\tau_{2n},\delta_{2n}],$$

and that at each stage the subalgebras are graded by total degree and the automorphisms respect the grading. Hence, C is Auslander-regular and CM by repeated applications of [19, Lemma]. It now follows that A is Auslander–Gorenstein and CM, by [29, Lemma 4.4]. In fact, A is Auslander-regular, since $\operatorname{gl.dim} A \le 2n < \infty$, by [24, Theorem 7.5.3]. □

In order to apply Theorem 1.6 to quantized Weyl algebras, we need to prove normal separation. Our method involves localizations in which we invert certain of the z_i, and it would be most convenient if we could reorder the z_i so that the ones to be inverted all occur at the beginning of the list. However, we can only partially achieve this aim – we are unable to bring a z_i that we wish to invert to the front of the list unless z_{i-1} is also to be inverted. Thus, we can move to the front all such z_i, resulting in a new ordering in which all the z_i from an initial string are to be inverted, while those z_i later in the list which are to be inverted are flanked on each side by z_j's which we do not invert. It is these "isolated" z_i which we are unable to move forward. The following lemma gives the technical details of this procedure.

Lemma 3.7. *Given any subset $M \subseteq \{1,\dots,n\}$, there exist a permutation $\pi \in S_n$ with corresponding permutation matrix Π and a k-algebra isomorphism*

$$\phi : A_n^{Q\Pi,\Pi^{-1}\Gamma\Pi}(k)\left[\mathscr{Z}^{-1}_{\pi^{-1}(M)}\right] \to A_n^{Q,\Gamma}(k)[\mathscr{Z}_M^{-1}]$$

such that

(a) $\pi^{-1}(M) = \{1,2,\dots,m_0\} \cup \{m_1,\dots,m_t\}$ *where* $m_s \ge m_{s-1}+2$ *for* $s = 1,\dots,t$.

(b) *Each* $\phi(\widehat{y}_i) = y_{\pi(i)}$ *and* $\phi(\widehat{x}_i) = u_i x_{\pi(i)}$, *where the* $u_i \in \mathscr{Z}_M^{-1}\mathscr{Z}_M$ *and* $\widehat{x}_1, \widehat{y}_1, \dots, \widehat{x}_n, \widehat{y}_n$ *are the canonical generators of* $A_n^{Q\Pi,\Pi^{-1}\Gamma\Pi}(k)$.

Proof. It suffices to show that if there is some $r \in \{3,\dots,n\}$ such that $r, r-1 \in M$ while $r-2 \notin M$, then there exists a k-algebra isomorphism ϕ as above, where $\pi = (r, r-1, \dots, 1)$. Let Π be the permutation matrix with entries $\pi_{ij} = \delta_{i,\pi(j)}$, and set

$$(\widehat{q}_1,\dots,\widehat{q}_n) = Q\Pi = (q_r, q_1, \dots, q_{r-1}, q_{r+1}, \dots, q_n),$$

and $(\widehat{\gamma}_{ij}) = \Pi^{-1}\Gamma\Pi = (\gamma_{\pi(i),\pi(j)})$. Write $\widehat{B}$ and B for the algebras $A_n^{Q\Pi,\Pi^{-1}\Gamma\Pi}(k)[\mathscr{Z}^{-1}_{\pi^{-1}(M)}]$ and $A_n^{Q,\Gamma}(k)[\mathscr{Z}_M^{-1}]$, and label the canonical generators for $\widehat{B}$ as $\widehat{x}_i$, $\widehat{y}_i$, $\widehat{z}_i^{-1}$ where $\widehat{z}_i = \widehat{x}_i\widehat{y}_i - \widehat{y}_i\widehat{x}_i$ and $\widehat{z}_i^{-1} \in \widehat{B}$ for $i \in \pi^{-1}(M)$.

We begin by identifying elements $\tilde{x}_i, \tilde{y}_i \in B$ satisfying the same relations as the $\widehat{x}_i, \widehat{y}_i$. First, set $\tilde{y}_i = y_{\pi(i)}$ for all i, and note that

$$\tilde{y}_i\tilde{y}_j = y_{\pi(i)}y_{\pi(j)} = \gamma_{\pi(i)\pi(j)}y_{\pi(j)}y_{\pi(i)} = \widehat{\gamma}_{ij}\tilde{y}_j\tilde{y}_i$$

for all i,j. Next, set $\tilde{z}_1 = z_{r-1}^{-1} z_r$ and

$$\tilde{x}_i = \begin{cases} z_{r-1}^{-1} x_r & (i = 1), \\ \tilde{z}_1 x_{i-1} & (i = 2,\dots,r), \\ x_i & (i = r+1,\dots,n). \end{cases}$$

Since z_{r-1} commutes with x_r and y_r, we have

$$\tilde{x}_1 \tilde{y}_1 - \widehat{q}_1 \tilde{y}_1 \tilde{x}_1 = z_{r-1}^{-1}(x_r y_r - q_r y_r x_r) = z_{r-1}^{-1} z_{r-1} = 1$$
$$\tilde{x}_1 \tilde{y}_1 - \tilde{y}_1 \tilde{x}_1 = z_{r-1}^{-1}(x_r y_r - y_r x_r) = z_{r-1}^{-1} z_r = \tilde{z}_1;$$

in particular, the notation $\tilde{z}_1$ is justified. Observe that $\tilde{z}_1$ commutes with x_i and y_i for all $i \neq r$, while $\tilde{z}_1 x_r = q_r^{-1} x_r \tilde{z}_1$ and $\tilde{z}_1 y_r = q_r y_r \tilde{z}_1$.

For $i = 2,\dots,r$, we now have

$$\tilde{x}_i \tilde{y}_i - \widehat{q}_i \tilde{y}_i \tilde{x}_i = \tilde{z}_1(x_{i-1} y_{i-1} - q_{i-1} y_{i-1} x_{i-1}) = \tilde{z}_1 \left(1 + \sum_{j=1}^{i-2} (q_j - 1) y_j x_j\right)$$
$$= \tilde{z}_1 + \sum_{l=2}^{i-1} (\widehat{q}_l - 1) \tilde{y}_l \tilde{x}_l = 1 + \sum_{l=1}^{i-1} (\widehat{q}_l - 1) \tilde{y}_l \tilde{x}_l.$$

If $\tilde{z}_i := \tilde{x}_i \tilde{y}_i - \tilde{y}_i \tilde{x}_i$, then, similarly, $\tilde{z}_i = \tilde{z}_1(x_{i-1} y_{i-1} - y_{i-1} x_{i-1}) = \tilde{z}_1 z_{i-1}$. In particular, $\tilde{z}_r = \tilde{z}_1 z_{r-1} = z_{r-1}^{-1} z_r z_{r-1} = z_r$. Hence, for $i = r+1,\dots,n$ we have

$$\tilde{x}_i \tilde{y}_i - \widehat{q}_i \tilde{y}_i \tilde{x}_i = x_i y_i - q_i y_i x_i = 1 + \sum_{l=1}^{i-1} (q_l - 1) y_l x_l = z_r + \sum_{l=r+1}^{i-1} (q_l - 1) y_l x_l$$
$$= \tilde{z}_r + \sum_{l=r+1}^{i-1} (\widehat{q}_l - 1) \tilde{y}_l \tilde{x}_l = 1 + \sum_{l=1}^{i-1} (\widehat{q}_l - 1) \tilde{y}_l \tilde{x}_l,$$

while $\tilde{z}_i := \tilde{x}_i \tilde{y}_i - \tilde{y}_i \tilde{x}_i = x_i y_i - y_i x_i = z_i$. Note that for all i, the element $\tilde{z}_i$ equals a unit of B times $z_{\pi(i)}$. Thus, $\tilde{z}_i^{-1} \in B$ for all $i \in \pi^{-1}(M)$.

The next step is to check the relations involving $\tilde{x}_i \tilde{x}_j$ and $\tilde{x}_i \tilde{y}_j$, which is easy but for the number of cases involved. We shall write out the details for the relations involving $\tilde{x}_i \tilde{x}_j$ and leave the others to the reader.

$$\tilde{x}_1 \tilde{x}_j = z_{r-1}^{-1} x_r \tilde{z}_1 x_{j-1} = q_r z_{r-1}^{-1} \tilde{z}_1 x_r x_{j-1} = q_r q_{j-1}^{-1} \gamma_{j-1,r}^{-1} z_{r-1}^{-1} \tilde{z}_1 x_{j-1} x_r$$
$$= q_r \gamma_{r,j-1} \tilde{z}_1 x_{j-1} z_{r-1}^{-1} x_r = \widehat{q}_1 \widehat{\gamma}_{1j} \tilde{x}_j \tilde{x}_1 \quad (1 < j \leq r),$$

$$\tilde{x}_1 \tilde{x}_j = z_{r-1}^{-1} x_r x_j = q_r \gamma_{rj} z_{r-1}^{-1} x_j x_r = q_r \gamma_{rj} x_j z_{r-1}^{-1} x_r = \widehat{q}_1 \widehat{\gamma}_{1j} \tilde{x}_j \tilde{x}_1 \quad (r < j),$$

$$\tilde{x}_i \tilde{x}_j = \tilde{z}_1 x_{i-1} \tilde{z}_1 x_{j-1} = q_{i-1} \gamma_{i-1,j-1} \tilde{z}_1 x_{j-1} \tilde{z}_1 x_{i-1} = \widehat{q}_i \widehat{\gamma}_{ij} \tilde{x}_j \tilde{x}_i \quad (1 < i < j \leq r),$$

$$\tilde{x}_i \tilde{x}_j = \tilde{z}_1 x_{i-1} x_j = q_{i-1} \gamma_{i-1,j} x_j \tilde{z}_1 x_{i-1} = \widehat{q}_i \widehat{\gamma}_{ij} \tilde{x}_j \tilde{x}_i \quad (1 < i \leq r < j),$$

$$\tilde{x}_i \tilde{x}_j = x_i x_j = q_i \gamma_{ij} x_j x_i = \widehat{q}_i \widehat{\gamma}_{ij} \tilde{x}_j \tilde{x}_i \quad (r < i < j).$$

Thus $\tilde{x}_i\tilde{x}_j = \widehat{q}_i\widehat{\gamma}_{ij}\tilde{x}_j\tilde{x}_i$ for all $i < j$. Similarly,

$$\tilde{x}_i\tilde{y}_j = \begin{cases} \widehat{\gamma}_{ji}\tilde{y}_j\tilde{x}_i & (i < j) \\ \widehat{q}_j\widehat{\gamma}_{ji}\tilde{y}_j\tilde{x}_i & (i > j). \end{cases}$$

Therefore there exists a k-algebra homomorphism $\phi : \widehat{B} \to B$ such that $\phi(\widehat{x}_i) = \tilde{x}_i$ and $\phi(\widehat{y}_i) = \tilde{y}_i$ for all i.

Obviously $y_i = \tilde{y}_{\pi^{-1}(i)} \in \phi(\widehat{B})$ for all i, and $x_i = \tilde{x}_i \in \phi(\widehat{B})$ for all $i > r$. Further, $z_i = \tilde{z}_i \in \phi(\widehat{B})$ for $i \geq r$, and since $\widehat{z}_1^{-1} \in \widehat{B}$ (because $1 \in \pi^{-1}(M)$) we have $z_i = \tilde{z}_1^{-1}\tilde{z}_{i+1} \in \phi(\widehat{B})$ for $i < r$. Consequently, $x_i = \tilde{z}_1^{-1}\tilde{x}_{i+1} \in \phi(\widehat{B})$ for $i = 1,\dots,r-1$ and $x_r = z_{r-1}\tilde{x}_1 \in \phi(\widehat{B})$. For $i \in M \cap \{1,\dots,r-1\}$, we have $z_i^{-1} = \tilde{z}_{i+1}^{-1}\tilde{z}_1 \in \phi(\widehat{B})$ because $i+1 \in \pi^{-1}(M)$, while for $i \in M \cap \{r,\dots,n\}$ we have $z_i^{-1} = \tilde{z}_i^{-1} \in \phi(\widehat{B})$ because $i \in \pi^{-1}(M)$ (recall that $r = \pi^{-1}(r-1) \in \pi^{-1}(M)$). Therefore $\phi(\widehat{B}) = B$.

Since B and $\widehat{B}$ are noetherian domains with the same finite GK-dimension (Lemma 3.4), the kernel of ϕ must be zero (cf. [16, Proposition 3.15 or Corollary 3.16] or [24, Corollary 8.3.6(ii)]). Therefore ϕ is an isomorphism. □

3.8. We next associate with each prime P in a quantized Weyl algebra A a localization $B_P = (A/I_P)[\mathscr{E}_P^{-1}]$ where I_P is a polynormal ideal of A and $\mathscr{E}_P$ is an Ore set of elements normal modulo I_P; Lemma 3.7 provides a key step in analyzing the structure of this algebra. Under the assumption that none of the q_i is a root of unity, we shall prove that $\operatorname{spec} B_P$ is centrally separated, and then normal separation for $\operatorname{spec} A$ follows readily.

There are only finitely many of the pairs $(I_P, \mathscr{E}_P)$, and if we group the primes associated with each pair $(I_\bullet, \mathscr{E}_\bullet)$, we obtain a stratification of $\operatorname{spec} A$ analogous to the stratification of $\operatorname{spec} \mathscr{O}_q(SL_n(\mathbb{C}))$ into the disjoint sets $\operatorname{spec}_w \mathscr{O}_q(SL_n(\mathbb{C}))$ for $w \in S_n \times S_n$ used in [11, 2.8; 12, Corollary 1.3] (cf. [3, Theorem 1.10]). We do not pursue the analogy here, except to point out the parallel with the proof of normal separation of $\operatorname{spec} \mathscr{O}_q(SL_n(\mathbb{C}))$ given in [3, Proposition 1.7, Theorem 5.8].

3.9. Let $A = A_n^{Q,\Gamma}(k)$ and $P \in \operatorname{spec} A$, and define the following index sets:

$$M = \{i \in \{1,\dots,n\} \mid z_i \notin P\},$$
$$M' = \{i \in \{1,\dots,n\} \mid y_i \notin P\},$$
$$M'' = \{i \in \{1,\dots,n\} \mid x_i \notin P\}.$$

Since $x_jy_j - y_jx_j = z_j$ for all j, we see that $M \subseteq M' \cap M''$. Further, x_j and y_j commute modulo $\langle z_j \rangle$, from which it follows (in view of the relations given in (3.1)) that x_j and y_j are normal modulo $\langle z_j \rangle$. Now set

$$I_P = \langle z_i \mid i \notin M \rangle + \langle y_i \mid i \notin M' \rangle + \langle x_i \mid i \notin M'' \rangle,$$

and note that I_P is a polynormal ideal of A contained in P. Next, let $\mathscr{E}_P$ denote the Ore set $\mathscr{Z}_M \mathscr{X}_{M'' \setminus M} \mathscr{Y}_{M' \setminus M}$ in A. Note that the x_i for $i \in M'' \setminus M$ and the y_i for $i \in M' \setminus M$ are normal modulo I_P, whence the elements of $\mathscr{E}_P$ are all normal modulo I_P. Since $\mathscr{E}_P$ is generated by normal elements not in P, it follows that the elements of $\mathscr{E}_P$ are all regular modulo P. Finally, set

$$B_P = (A/I_P)[\mathscr{E}_P^{-1}],$$

and use $a \mapsto \tilde{a}$ to denote the natural map $A \xrightarrow{\text{quo}} A/I_P \xrightarrow{\text{loc}} B_P$.

It can be shown that B_P is isomorphic to an iterated skew-Laurent algebra of the type described below (provided the $q_i \neq 1$). However, for our present purposes it suffices to exhibit B_P as a homomorphic image of such an algebra.

Proposition. *If all $q_i \neq 1$, then B_P is a homomorphic image of an iterated skew-Laurent algebra of the form*

$$B_r^{Q',\Gamma'}(k)[w_1^{\pm 1};\tau_1][w_2^{\pm 1};\tau_2]\cdots[w_v^{\pm 1};\tau_v],$$

where

(a) *Q' is a permutation of a subvector of Q, and Γ' is a permutation of the corresponding submatrix of Γ.*

(b) *Each τ_j is a k-algebra automorphism such that $\tau_j(B_r^{Q',\Gamma'}(k)) = B_r^{Q',\Gamma'}(k)$.*

(c) *$\tau_j(w_l) \in k^\times w_l$ for $l < j$.*

Proof. Observe that B_P can be obtained from $A[\mathscr{X}_M^{-1}]$ by factoring out $I_P[\mathscr{X}_M^{-1}]$ and then localizing with respect to the image of $\mathscr{E}_P$. Hence, an isomorphic algebra results if we apply the change of variables for $A[\mathscr{X}_M^{-1}]$ given in Lemma 3.7. Thus, there is no loss of generality in assuming that $M = \{1,2,\ldots,r\} \cup \{m_1,\ldots,m_t\}$ where $m_1 \geq r+2$ and $m_s \geq m_{s-1}+2$ for $s = 2,\ldots,t$. In other words, $j-1$ and j cannot both belong to M for $j = r+1,\ldots,n$.

Set $B_r = B_r^{Q',\Gamma'}(k)$ where $Q' = (q_1,\ldots,q_r)$ and Γ' is the upper left $r \times r$ block of Γ. Then B_r is a subalgebra of $A[\mathscr{X}_M^{-1}]$, and we have a k-algebra homomorphism $\phi_r : B_r \to B_P$ such that $\phi_r(x_i) = \tilde{x}_i$ and $\phi_r(y_i) = \tilde{y}_i$ for $i = 1,\ldots,r$. Note that $\tilde{z}_1^{-1},\ldots,\tilde{z}_r^{-1}$ all lie in $\phi_r(B_r)$.

We next observe that for $j = r+1,\ldots,n$, there is a scalar $\alpha_j \in k^\times$ such that $\tilde{x}_j\tilde{y}_j = \alpha_j\tilde{y}_j\tilde{x}_j$. If $j \notin M$, then $\tilde{x}_j\tilde{y}_j - \tilde{y}_j\tilde{x}_j = \tilde{z}_j = 0$, and we take $\alpha_j = 1$. On the other hand, if $j \in M$, then $j \geq r+2$ and $j-1 \notin M$. In this case, $\tilde{x}_j\tilde{y}_j - q_j\tilde{y}_j\tilde{x}_j = \tilde{z}_{j-1} = 0$, and we take $\alpha_j = q_j$.

Now construct the following iterated skew polynomial ring:

$$B = B_r[u_{r+1};\rho_{r+1}][v_{r+1};\sigma_{r+1}]\cdots[u_n;\rho_n][v_n;\sigma_n],$$

where the ρ_j and σ_j are k-algebra automorphisms satisfying the rules below.

$$\begin{aligned}
&\rho_j(y_i) = \gamma_{ji}y_i, && \sigma_j(y_i) = q_i\gamma_{ij}y_i && (1 \leq i \leq r),\\
&\rho_j(u_i) = \gamma_{ji}u_i, && \sigma_j(u_i) = q_i\gamma_{ij}u_i && (r < i < j),\\
&\rho_j(x_i) = \gamma_{ij}x_i, && \sigma_j(x_i) = q_i^{-1}\gamma_{ji}x_i && (1 \leq i \leq r),\\
&\rho_j(v_i) = \gamma_{ij}v_i, && \sigma_j(v_i) = q_i^{-1}\gamma_{ji}v_i && (r < i < j),\\
& && \sigma_j(u_j) = \alpha_j u_j. &&
\end{aligned}$$

That automorphisms of this form exist follows from the fact that B_r has commuting automorphisms $\rho_{r+1}^\circ, \sigma_{r+1}^\circ, \ldots, \rho_n^\circ, \sigma_n^\circ$ satisfying the given rules; namely, ρ_j° equals conjugation by y_j and σ_j° equals conjugation by x_j.

We claim that ϕ_r extends to a k-algebra homomorphism $\phi : B \to B_P$ sending u_j to $\tilde{y}_j$ and v_j to $\tilde{x}_j$ for $j = r+1, \ldots, n$; this requires checking that the $\tilde{y}_j$ and $\tilde{x}_j$ satisfy the same relations as the u_j and v_j. For instance, for $i \le r < j$ we have $\tilde{y}_j\tilde{y}_i = \gamma_{ji}\tilde{y}_i\tilde{y}_j = \phi_r\rho_j(y_i)\tilde{y}_j$ and $\tilde{y}_j\tilde{x}_i = \gamma_{ij}\tilde{x}_i\tilde{y}_j = \phi_r\rho_j(x_i)\tilde{y}_j$, from which we see that $\tilde{y}_j\phi_r(b) = \phi_r\rho_j(b)\tilde{y}_j$ for all $b \in B_r$. Similarly, $\tilde{x}_j\phi_r(b) = \phi_r\sigma_j(b)\tilde{x}_j$ for all $b \in B_r$. Since the relations among $\tilde{y}_{r+1}, \tilde{x}_{r+1}, \ldots, \tilde{y}_n, \tilde{x}_n$ include those assigned to $u_{r+1}, v_{r+1}, \ldots, u_n, v_n$ (by construction), we conclude that there does exist a k-algebra homomorphism ϕ as described.

Finally, set

$$J = \langle u_i \mid i \in \{r+1, \ldots, n\} \setminus M' \rangle + \langle v_i \mid i \in \{r+1, \ldots, n\} \setminus M'' \rangle,$$
$$B' = (B/J)[u_i^{-1} \mid i \in M' \cap \{r+1, \ldots, n\}][v_i^{-1} \mid i \in M'' \cap \{r+1, \ldots, n\}];$$

then B' can be presented as an iterated skew-Laurent extension of B_r satisfying properties (b) and (c). By construction, $\tilde{y}_i = 0$ for $i \notin M'$ and $\tilde{x}_i = 0$ for $i \notin M''$, whence $J \subseteq \ker\phi$. We claim that $\tilde{y}_i$ is invertible in B_P for $i \in M' \cap \{r+1, \ldots, n\}$ and that $\tilde{x}_i$ is invertible in B_P for $i \in M'' \cap \{r+1, \ldots, n\}$.

Consider $i \in M' \cap \{r+1, \ldots, n\}$. If $i \notin M$, then $y_i \in \mathscr{E}_P$ and $\tilde{y}_i$ is invertible in B_P by construction. If $i \in M$, then $i \ge r+2$ and $i-1 \notin M$. In this case,

$$z_i = z_{i-1} + (q_i - 1)y_ix_i \equiv (q_i - 1)y_ix_i \pmod{I_P},$$

whence $\tilde{y}_i\tilde{x}_i = (q_i - 1)^{-1}\tilde{z}_i$. But $\tilde{z}_i$ is invertible in B_P because $i \in M$, and thus $\tilde{y}_i$ is too. A similar argument shows that $\tilde{x}_i$ is invertible in B_P for $i \in M'' \cap \{r+1, \ldots, n\}$, as claimed.

Therefore ϕ induces a k-algebra homomorphism $\phi' : B' \to B_P$, and we complete the proof by showing that ϕ' is surjective. Now $\tilde{y}_j, \tilde{x}_j \in \phi(B) \subseteq \phi'(B')$ for all j, and $\tilde{z}_j^{-1} \in \phi_r(B_r) \subseteq \phi'(B')$ for all $j \le r$. We must also show that the inverses of the remaining generators of $\mathscr{E}_P$ are contained in the image of ϕ'. For $i \in M' \setminus M$, we have $i > r$ and so $\tilde{y}_i^{-1} = \phi'(u_i^{-1})$. Similarly, $\tilde{x}_i^{-1} = \phi'(v_i^{-1})$ for $i \in M'' \setminus M$. Finally, consider $j \in M \cap \{r+1, \ldots, n\}$, and recall that $j \in M' \cap M''$. By construction, u_j and v_j are invertible in B', whence $\tilde{y}_j^{-1}, \tilde{x}_j^{-1} \in \phi'(B')$. Since $\tilde{z}_j = (q_j - 1)\tilde{y}_j\tilde{x}_j$ in this case (recall that $j - 1 \notin M$), we conclude that $\tilde{z}_j^{-1} \in \phi'(B')$, as desired.

Therefore ϕ' is surjective. □

Our final step in proving normal separation for quantized Weyl algebras is to show that the iterated skew-Laurent algebras appearing in Proposition 3.9 are polycentral. We do this under the hypothesis that none of the q_i is a root of unity. In that case, a result of Jordan [15, Theorem 3.2] shows that the algebras $B_r^{Q',\Gamma'}(k)$ are simple, and the following extension of Proposition 2.2 yields the desired polycentrality.

Proposition 3.10. *Let $A = B[x_1^{\pm 1}; \tau_1][x_2^{\pm 1}; \tau_2] \cdots [x_n^{\pm 1}; \tau_n]$ be an iterated skew-Laurent extension where B is a simple noetherian k-algebra, each τ_i is a k-algebra automorphism such that $\tau_i(B) = B$, and $\tau_i(x_j) \in k^\times x_j$ for $j < i$. Then A is polycentral.*

Proof. As in the proof of Proposition 2.2, we just need to show that for any ideals $I > J$ in A, there exists an element $u \in I \setminus J$ which is central modulo J.

Choose an element $u \in I \setminus J$ of minimum length, say length d. Then $u = b_1 x^{s_1} + b_2 x^{s_2} + \cdots + b_d x^{s_d}$ for some nonzero elements $b_h \in B$ and some distinct n-tuples $s_h \in \mathbb{Z}^n$. After replacing u by $u(x^{s_1})^{-1}$, we may assume that $s_1 = (0, \ldots, 0)$.

Since B is simple, there exist elements $e_j, f_j \in B$ such that $\sum_j e_j b_1 f_j = 1$. Set $u' = \sum_j e_j u f_j \in I$, and observe that $u' = 1 + b'_2 x^{s_2} + \ldots + b'_d x^{s_d}$ for some $b'_h \in B$. Consequently,

$$u - b_1 u' = (b_2 - b_1 b'_2) x^{s_2} + \ldots + (b_d - b_1 b'_d) x^{s_d}$$

is an element of I with length less than d, whence $u - b_1 u' \in J$. Since $u \notin J$, it follows that $u' \notin J$, and so we may replace u by u'. Thus, there is no loss of generality in assuming that $b_1 = 1$, that is, $u = 1 + b_2 x^{s_2} + \ldots + b_d x^{s_d}$.

Fix $t \in \{1, \ldots, n\}$. For $h = 2, \ldots, d$, note that $(b_h x^{s_h}) x_t$ and $x_t(b_h x^{s_h})$ both lie in $Bx^{s_h + e_t}$ where $e_t = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the tth position. Hence, $ux_t - x_t u$ has length less than d, and so $ux_t - x_t u \in J$. Similarly, for $b \in B$ we observe that $ub - bu$ has length less than d, whence $ub - bu \in J$. Therefore $u + J$ is central in A/J, as desired. □

3.11. Proposition 3.10 can be improved, if desired. First, it is not necessary to assume that B is simple, only that B is T-simple where T is the subgroup of $\operatorname{Aut} B$ generated by the restrictions of the τ_i. Second, one can prove a version analogous to the first part of Proposition 2.2. Namely, if

$$A_l = B[x_1^{\pm 1}; \tau_1] \cdots [x_l^{\pm 1}; \tau_l][x_{l+1}; \tau_{l+1}] \cdots [x_n; \tau_n],$$

and if B is T_l-simple where T_l is generated by the restrictions of $\tau_1, \ldots, \tau_l$ to B, then A_l is polynormal. We leave the details of these extensions to the reader.

Theorem 3.12. *Let $A = A_n^{Q,\Gamma}(k)$ as in* (3.1). *If none of the q_i is a root of unity, then* $\operatorname{spec} A$ *has normal separation.*

Proof. Consider distinct comparable prime ideals $P < P'$ in $\operatorname{spec} A$, and define I_P and $\mathscr{E}_P$ as in (3.9). Since the elements of $\mathscr{E}_P$ are regular and normal modulo P, if $P' \cap \mathscr{E}_P$ is nonempty we immediately obtain a nonzero normal element in P'/P. Hence, we may assume that $P' \cap \mathscr{E}_P = \emptyset$, and so we obtain distinct comparable primes $PB_P < P'B_P$ in the localization B_P.

Now B_P is a homomorphic image of an iterated skew-Laurent algebra of the form

$$B_r^{Q',\Gamma'}(k)[w_1^{\pm 1}; \tau_1][w_2^{\pm 1}; \tau_2] \cdots [w_v^{\pm 1}; \tau_v]$$

as in Proposition 3.9. Since none of the q_i is a root of unity, $B_r^{Q',\Gamma'}(k)$ is simple [15, Theorem 3.2]. Therefore it follows from Proposition 3.10 that B_P is polycentral. In particular, there exists a nonzero central element $c \in B_P/PB_P$ which lies in $P'B_P/PB_P$. Then $c = c'e^{-1}$ for some $c' \in P' \setminus P$ and some $e \in \mathscr{E}_P$, and since e is regular and normal modulo P, we conclude that c' is normal modulo P. Thus we have normal separation in all cases. □

Theorem 3.13. *Let $Q = (q_1, \ldots, q_n) \in (k^\times)^n$. If none of the q_i is a root of unity, then the quantized Weyl algebra $A_n^{Q,\Gamma}(k)$ is catenary, and Tauvel's height formula holds in $A_n^{Q,\Gamma}(k)$.*

Proof. All the hypotheses of Theorem 1.6 are satisfied by $A_n^{Q,\Gamma}(k)$, in view of Proposition 3.4 and Theorems 3.6, 3.12. □

4. Other quantum algebras

This short final section is devoted to two important algebras for which an extensive structure theory is already known – the quantum general linear group and the quantized enveloping algebra of a maximal nilpotent subalgebra of a semisimple Lie algebra. We begin by discussing the coordinate ring of the quantum general linear group. While it is not hard to obtain the Auslander-regular and Cohen–Macaulay properties for general multiparameter versions of this algebra, at present normal separation is only known in the one-parameter case over the complex field. We ignore the case in which the parameter q is a root of unity, since then the quantum general linear group satisfies a polynomial identity, and both normal separation and catenarity follow from standard PI theory. Normal separation in the case that q is not a root of unity was derived by Brown and the first author [3] from Hodges and Levasseur's fundamental work on the quantum special linear group [11, 12].

4.1. We define the one-parameter coordinate ring of quantum matrices over a field k as in [28, p. 145]; here the parameter q is any nonzero element of k. This is the k-algebra $\mathcal{O}_q(M_n(k))$ generated by elements x_{ij} for $i, j = 1, \ldots, n$ subject to the following relations:

$$x_{ij}x_{lj} = q^2 x_{lj}x_{ij} \quad (i < l),$$
$$x_{ij}x_{im} = q^2 x_{im}x_{ij} \quad (j < m),$$
$$x_{im}x_{lj} = x_{lj}x_{im} \quad (i < l \text{ and } j < m),$$
$$x_{ij}x_{lm} - x_{lm}x_{ij} = (q^2 - q^{-2})x_{im}x_{lj} \quad (i < l \text{ and } j < m).$$

It is well known that $\mathcal{O}_q(M_n(k))$ is an iterated skew polynomial extension of k (cf. [9, 3.1]) and hence a noetherian domain, and that this algebra has GK-dimension n^2 [25, Theorem 3.5.1].

Recall that the coordinate ring of the quantum general linear group, $\mathcal{O}_q(GL_n(k))$, is formed from $\mathcal{O}_q(M_n(k))$ by inverting the "quantum determinant"

$$\det_q X = \sum_{\sigma \in S_n} (-q^{-2})^{l(\sigma)} x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)},$$

which is a central element in $\mathcal{O}_q(M_n(k))$. In particular, $\mathcal{O}_q(GL_n(k))$ is a noetherian domain, with GK-dimension n^2 (use [16, Proposition 4.2] or [24, Proposition 8.2.13]). The coordinate ring of the quantum special linear group, $\mathcal{O}_q(SL_n(k))$, is the factor ring $\mathcal{O}_q(M_n(k))/\langle \det_q X - 1 \rangle$. The homological properties we require of $\mathcal{O}_q(GL_n(k))$ can be obtained from those determined for $\mathcal{O}_q(SL_n(k))$ by Levasseur and Stafford [19], as follows.

Theorem 4.2. *The coordinate ring $\mathcal{O}_q(GL_n(k))$ of the quantum general linear group is Auslander-regular and Cohen–Macaulay.*

Proof. First, $\mathcal{O}_q(SL_n(k))$ is Auslander-regular and CM by [19, Corollary]. By [19, Lemma], these properties carry over to the polynomial ring $\mathcal{O}_q(SL_n(k))[z, z']$ (where z and z' are central indeterminates). Then, since $zz' - 1$ is a central regular element of $\mathcal{O}_q(SL_n(k))[z, z']$, [19, Lemma] shows that the factor ring $\mathcal{O}_q(SL_n(k))[z, z']/\langle zz' - 1 \rangle$ is Auslander–Gorenstein and CM. The latter algebra is isomorphic to the Laurent polynomial ring $\mathcal{O}_q(SL_n(k))[z, z^{-1}]$, and hence to $\mathcal{O}_q(GL_n(k))$ [19, Proposition]. Therefore $\mathcal{O}_q(GL_n(k))$ is Auslander–Gorenstein and CM. On the other hand,

$$\operatorname{gl.dim} \mathcal{O}_q(GL_n(k)) \leq \operatorname{gl.dim} \mathcal{O}_q(M_n(k)) \leq n^2 < \infty,$$

because of the structure of $\mathcal{O}_q(M_n(k))$ as an iterated skew polynomial extension of k [24, Theorem 7.5.3]. Thus $\mathcal{O}_q(GL_n(k))$ is actually Auslander-regular. □

Theorem 4.3. *If $q \in \mathbb{C}^\times$ is not a root of unity, then* $\operatorname{spec} \mathcal{O}_q(GL_n(\mathbb{C}))$ *has normal separation.*

Proof. [3, 6.14]. □

4.4. Brown and the first author have conjectured that not only $\operatorname{spec} \mathcal{O}_q(GL_n(k))$ but also $\operatorname{spec} \mathcal{O}_q(M_n(k))$ has normal separation, at least for algebraically closed fields k. (In particular, this can be easily checked in the case $n = 2$.) Since $\mathcal{O}_q(M_n(k))$ is Auslander-regular and Cohen–Macaulay [19, proof of Corollary], it would then follow that $\mathcal{O}_q(M_n(k))$ is catenary.

Theorem 4.5. *If $q \in \mathbb{C}^\times$ is not a root of unity, then $\mathcal{O}_q(GL_n(\mathbb{C}))$ is catenary, and Tauvel's height formula holds in $\mathcal{O}_q(GL_n(\mathbb{C}))$.*

Proof. See Theorems 4.2, 4.3, and 1.6. □

Theorem 4.5 of course holds also for $\mathcal{O}_q(SL_n(\mathbb{C}))$, which is a factor of $\mathcal{O}_q(GL_n(\mathbb{C}))$ by a height 1 prime ideal.

Normal separation is known for the quantum coordinate rings $\mathcal{O}_q(G)$ of connected semisimple algebraic groups over $\mathbb{C}$ (see [3, Theorem 5.8]). We conjecture that these algebras are also Auslander-regular and Cohen–Macaulay, hence catenary.

4.6. Malliavin has recently proved that the quantized enveloping algebra of a maximal nilpotent subalgebra of the simple Lie algebra of type B_2 over an algebraically closed field of characteristic zero is catenary [21, Théorème 2]. In fact, this conclusion holds for the quantized enveloping algebra $U_q(\boldsymbol{n}^+)$ of a maximal nilpotent subalgebra $\boldsymbol{n}$ of an arbitrary finite dimensional semisimple complex Lie algebra $\boldsymbol{g}$. The normal elements needed to apply our method exist by work of Caldero [4], and the requisite homological properties follow from results of Ringel [26]. Generators and relations for the quantized enveloping algebra of $\boldsymbol{g}$ as defined by Lusztig can be found in [20]; we do not recall them here, but only those required for the subalgebra corresponding to $U_q(\boldsymbol{n}^+)$.

Let (a_{ij}) be the Cartan matrix of $\boldsymbol{g}$; this is an $n \times n$ integer matrix (where n is the rank of $\boldsymbol{g}$), and there exist relatively prime positive integers d_i such that the matrix $(d_i a_{ij})$ is symmetric. Let $\mathbb{Q}(q)$ be a rational function field over $\mathbb{Q}$ in an indeterminate q, set $v = q^2$, and define $\boldsymbol{U}^+$ to be the $\mathbb{Q}(v)$-algebra with generators $E_1, \dots, E_n$ subject to the relations

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0$$

for all $i \neq j$, where $\binom{1-a_{ij}}{s}_{d_i}$ is a v^{d_i}-binomial coefficient. These coefficients are defined as follows:

$$\binom{r}{s}_d = \frac{[r]_d^!}{[s]_d^![r-s]_d^!}, \quad \text{where} \quad [r]_d^! = \prod_{t=1}^{r} \frac{v^{dt} - v^{-dt}}{v^d - v^{-d}}.$$

The quantized enveloping algebra $U_q(\boldsymbol{n}^+)$, finally, may be defined as $\boldsymbol{U}^+ \otimes_{\mathbb{Q}(v)} \mathbb{C}(q)$. (The algebra $U_q(\boldsymbol{n}^+)$ is denoted U^+ in [4, Section 1.2]. In [26], on the other hand, the symbol U^+ denotes the algebra we have labelled $\boldsymbol{U}^+$.)

Theorem 4.7. *The quantized enveloping algebra $U_q(\boldsymbol{n}^+)$ is an affine noetherian $\mathbb{C}(q)$-algebra with finite GK-dimension. Moreover, $U_q(\boldsymbol{n}^+)$ is an Auslander-regular, Cohen–Macaulay domain.*

Proof. By definition, $U_q(\boldsymbol{n}^+)$ is an affine $\mathbb{C}(q)$-algebra.

Ringel has shown in [26, Sections 4,5] that the algebra $\boldsymbol{U}^+$ is an iterated skew polynomial ring of the form

$$\boldsymbol{U}^+ = \mathbb{Q}(v)[X_1][X_2; \tau_2, \delta_2] \cdots [X_m; \tau_m, \delta_m]$$

where the τ_j are $\mathbb{Q}(v)$-algebra automorphisms such that $\tau_j(X_i) \in v^{\mathbb{Z}}X_i$ for $i < j$, and the δ_j are $\mathbb{Q}(v)$-linear τ_j-derivations such that $\delta_j(X_i)$ (for $i < j$) is a linear combination of monomials in $X_{i+1},\ldots,X_{j-1}$. Further, U^+ has a natural $\mathbb{Z}^n$-grading under which the X_j are homogeneous elements. For $l \in \mathbb{Z}$, let U_l^+ be the sum of the homogeneous components of U^+ over those n-tuples $(l_1,\ldots,l_n) \in \mathbb{Z}^n$ for which $l_1+\ldots+l_n = l$. The decomposition $U^+ = \bigoplus_l U_l^+$ is a $\mathbb{Z}$-grading of U^+ in which the X_j are homogeneous of positive degree. Each iteration $\mathbb{Q}(v)\langle X_1,\ldots,X_{j-1}\rangle$ is then a connected graded $\mathbb{Q}(v)$-algebra, and the automorphism τ_j respects the grading on $\mathbb{Q}(v)\langle X_1,\ldots,X_{j-1}\rangle$.

All the $\mathbb{Q}(v)$-algebra structure just described for U^+ carries over to corresponding $\mathbb{C}(q)$-algebra structure for $U_q(\mathfrak{n}^+)$. Hence, $U_q(\mathfrak{n}^+)$ is a noetherian domain, and iterated application of [19, Lemma] shows that $U_q(\mathfrak{n}^+)$ is Auslander-regular and CM.

It remains to show that $\mathrm{GKdim}(U_q(\mathfrak{n}^+))$ is finite. This follows from iterated application of a slight enhancement of [16, Proposition 3.5]. Namely, suppose that A is an affine algebra over a field k and $B = A[x;\tau,\delta]$ is a skew polynomial ring constructed from a k-algebra automorphism τ and a k-linear τ-derivation δ. If A has a finite dimensional generating subspace which is τ-stable, then $\mathrm{GKdim}(B) = \mathrm{GKdim}(A) + 1$. We leave the verification of this equality to the reader. □

Theorem 4.8. *The quantized enveloping algebra $U_q(\mathfrak{n}^+)$ is catenary, and Tauvel's height formula holds in $U_q(\mathfrak{n}^+)$.*

Proof. Caldero has shown that every ideal of $U_q(\mathfrak{n}^+)$ has a normalizing sequence of generators [4, Corollaire 3.2]. Normal separation in $\operatorname{spec} U_q(\mathfrak{n}^+)$ follows immediately, and then Theorems 4.7 and 1.6 yield the desired conclusions. □

Notes added in proof (January 1996).

(a) Further studies of prime ideals in quantized Weyl algebras may be found in papers of Akhavizadegan and Jordan [31] and Rigal [33].

(b) Theorem 1.6 has been applied by Oh to obtain catenarity in the quantum coordinate rings of symplectic and Euclidean spaces [32].

(c) For graded algebras, several of the hypotheses of Theorem 1.6 are redundant, due to a recent result of Zhang [34]: In a connected graded noetherian algebra of finite injective dimension, normal separation implies the Auslander–Gorenstein and Cohen–Macaulay properties, as well as finiteness of the Gelfand–Kirillov dimension.

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